DIMENSIONAL ANALYSIS AND SIMILARITY[1]

INTRODUCTION

In Fluid Mechanics a lot of the data we use is obtained experimentally or computationally. There are some situations where it is impossible to obtain the required data at the required scale. For example how would you measure the drag coefficient on a block of flats in central London? It is obviously far easier if we can test at a reduced scale, but we need to know how to scale appropriately. If we produce a 1/1000 scale model, what air speed do we need for it to be equivalent to the air speed in the full size case. Is there anything else we need to take into account? Or if I do some testing for pressure difference in a rough pipe carrying oil, what do I need to do to work out the pressure difference for a smooth pipe carrying water?

The approach adopted in Fluids (among others) is to make use of non-dimensional groups. Some of these have already been seen in this and other modules (Reynolds number, Drag coefficient, Lift coefficient etc) and in this section further groups will be introduced.

DIMENSIONS AND UNITS

The four basic *dimensions* that govern most problems in fluid mechanics are mass, M, length, L, time, T and temperature, θ. There are additional dimensions that are required in some other fields and problems (for example electrical and magnetic charges, magneto hydrodynamics). All other quantities can be found as combinations of these fundamental dimensions, so for example velocity has the dimensions [L/T] or [LT^{-1}]. The units of velocity vary according to what system you are using (metres per second, feet per second, miles per hour etc) but the dimensions are always the same, [LT^{-1}].

Some standard thermofluids quantities and their dimensions are given in Table 1 (for more details see [1]).

[1] A lot of the material in this chapter is drawn from [1]

Table 1: Dimensions of standard thermofluids quantities [1]

Quantity	Symbol	Dimensions	
		$MLT\Theta$	$FLT\Theta$
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	$\dot{m}$	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ, τ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$\dot{\epsilon}$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω, Ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Thermal expansion coefficient	β	Θ^{-1}	Θ^{-1}

A non-dimensional group does not have any dimensions (or units!). Think about Reynolds number for example

$$Re = \frac{\rho v d}{\mu}$$

According with Table 1, we have:

Quantity	SI unit	Dimensions
ρ, density	kg/m^3	ML^{-3}
v, velocity	m/s	LT^{-1}
D, length or diameter	m	L
μ, viscosity	kg/ms	$ML^{-1}T^{-1}$

So

$$[Re] = \frac{[ML^{-3}][LT^{-1}][L]}{[ML^{-1}T^{-1}]} = [\,]$$

STANDARD DIMENSIONLESS GROUPS IN FLUIDS

There are many standard non-dimensional groups in thermofluids all of them defined by the governing equations and boundary conditions. These are groups that characterised the dynamics of the flow problems. You would not normally develop a different group to one of these unless you need to consider different physical processes affecting the fluid flow problem in consideration. These are some of the most common.

Reynolds number, $Re = \frac{\rho U L}{\mu}$. Usually regarded as non-dimensional **flowrate**. Reynolds number is the most important parameter in fluid mechanics. It will almost always be one of the dimensionless groups you find, it is important in flows with and without a free surface and should only be neglected where there are not high velocity gradients. (Named after Osbourne Reynolds).

Froude number, $Fr = \frac{U^2}{gL}$. Usually regarded as non-dimensional **gravity**. Froude number contains g, gravitational acceleration, and is dominant in free surface flows. It is unimportant if there is no free surface but very important for boats, waves etc. (Named after William Froude).

Weber number, $We = \frac{\rho U^2 L}{\sigma}$. Where σ is the surface tension coefficient with unit M/T^2. Weber number is often only important if it is of order 1 or less – usually required for droplets, capillary flow, ripple waves and very small hydraulic models. (Named after Moritz Weber).

Euler number, $Eu = \frac{p}{\rho U^2}\, or\, \frac{\Delta p}{\rho U^2}$. The Euler number is a non-dimensional **pressure** or **pressure difference**. Euler number based on atmospheric pressure is rarely important except where the pressure is low enough that vapour can form. This is called cavitation and the **Cavitation number** is $Ca = \frac{p_a - p_v}{\rho U^2}$ where p_v is the vapour pressure. (Named after Leonhard Euler)

Mach number, $Ma = \frac{U}{a}$. This is essentially non-dimensional **velocity for compressible flows** (transonic upwards usually). In high speed gas flow compressibility effects become extremely important and gas properties have to be obtained from an equation of state such as the perfect gas law. The Mach number is the ratio of velocity to the speed of sound at the same gas conditions in that medium. (Named after Ernst Mach)

Strouhal number, $St = \frac{fL}{U}\, or\, \frac{\omega L}{U}$. Usually regarded as non-dimensional **frequency.** There are a number of flows that are oscillatory, for example the flow behind a circular cylinder forms a vortex shedding pattern with a regular frequency of shedding. (Named after Vincenc Strouhal.)

Others we have already met include lift and drag coefficients, C_D and C_L, roughness ratio (external flows), $\frac{L}{\epsilon}$ or relative roughness (pipe and duct flow), $\frac{k}{d}$ and loss coefficients, $K = \frac{\Delta p}{\frac{1}{2}\rho U^2}$ (Euler). Although these latter can be seen as just a particular sort of Euler number.

White [1] includes a table of dimensionless groups which is reproduced on the next page for your reference (Table 2).

Table 2: Dimensionless groups in fluid mechanics

Parameter	Definition	Qualitative ratio of effects	Importance
Reynolds number	$\text{Re} = \frac{\rho U L}{\mu}$	$\frac{\text{Inertia}}{\text{Viscosity}}$	Almost always
Mach number	$\text{Ma} = \frac{U}{a}$	$\frac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude number	$\text{Fr} = \frac{U^2}{gL}$	$\frac{\text{Inertia}}{\text{Gravity}}$	Free-surface flow
Weber number	$\text{We} = \frac{\rho U^2 L}{\Upsilon}$	$\frac{\text{Inertia}}{\text{Surface tension}}$	Free-surface flow
Rossby number	$\text{Ro} = \frac{U}{\Omega_{\text{earth}} L}$	$\frac{\text{Flow velocity}}{\text{Coriolis effect}}$	Geophysical flows
Cavitation number (Euler number)	$\text{Ca} = \frac{p - p_v}{\rho U^2}$	$\frac{\text{Pressure}}{\text{Inertia}}$	Cavitation
Prandtl number	$\text{Pr} = \frac{\mu c_p}{k}$	$\frac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Eckert number	$\text{Ec} = \frac{U^2}{c_p T_0}$	$\frac{\text{Kinetic energy}}{\text{Enthalpy}}$	Dissipation
Specific-heat ratio	$k = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$	Compressible flow
Strouhal number	$\text{St} = \frac{\omega L}{U}$	$\frac{\text{Oscillation}}{\text{Mean speed}}$	Oscillating flow
Roughness ratio	$\frac{\epsilon}{L}$	$\frac{\text{Wall roughness}}{\text{Body length}}$	Turbulent, rough walls
Grashof number	$\text{Gr} = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Rayleigh number	$\text{Ra} = \frac{\beta \Delta T g L^3 \rho^2 c_p}{\mu k}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Temperature ratio	$\frac{T_w}{T_0}$	$\frac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer
Pressure coefficient	$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aerodynamics, hydrodynamics
Lift coefficient	$C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Drag coefficient	$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Friction factor	$f = \frac{h_f}{(V^2/2g)(L/d)}$	$\frac{\text{Friction head loss}}{\text{Velocity head}}$	Pipe flow
Skin friction coefficient	$c_f = \frac{\tau_{\text{wall}}}{\rho V^2/2}$	$\frac{\text{Wall shear stress}}{\text{Dynamic pressure}}$	Boundary layer flow

USES OF DIMENSIONAL ANALYSIS

The most obvious are where dimensional analysis is useful is in scaling experiments. If I want to know how an aeroplane wing will perform it is very expensive to build one and test it. In addition it is quite difficult to instrument a full scale wing at 10,000m above the ground moving at 500 mph. Scale testing is used extensively in engineering and dimensional analysis is one of the tools to help us scale all the experimental variables appropriately.

Another of the areas where dimensional analysis is particularly useful is in the reduction the number of experiments that must be carried out in order to characterise a system.

For example, suppose I want to investigate the torque on an enclosed rotating cone (as illustrated in Figure 1) where we vary the liquid surrounding the cone. I take data using water and a couple of other liquids (oils), Aeroshell 390 and Velocite No 4 for example. The results might look as shown in Figure 2. There is a family of curves that look similar but if I want to know what torque I will obtain using the same fluid but a cone of different diameter, or the same cone but aσ erent fluid I don't have the information I need on this graph.

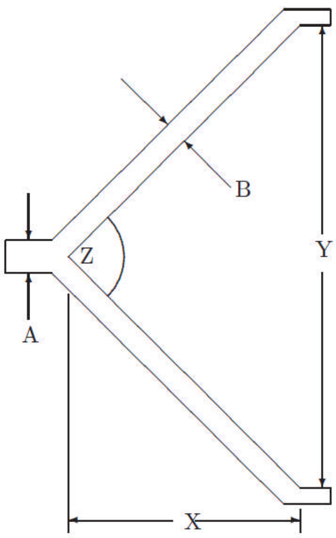

Figure 1: Enclosed rotating cone [2]

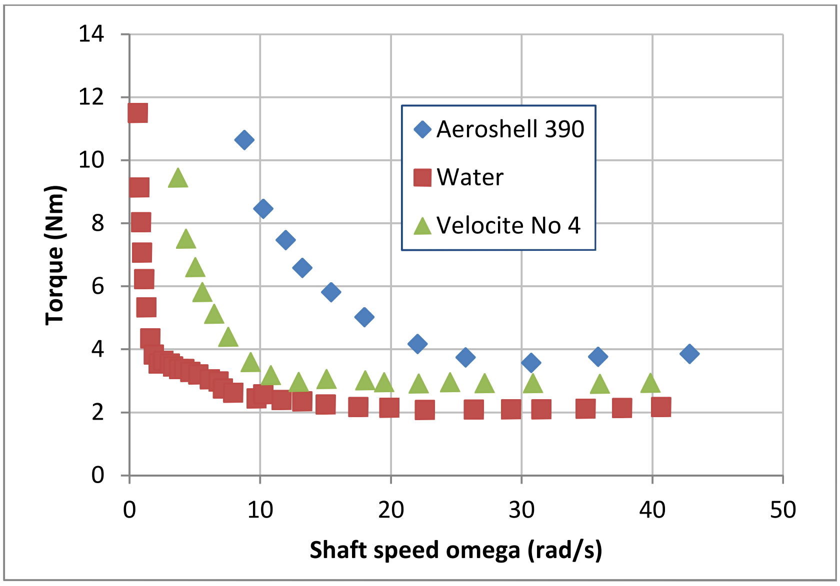

Figure 2: Graph showing variation of torque with shaft speed

It can be considered that torque is a function of only a few variables (found from the governing equations and definition of the problem): the speed of rotation, the diameter of the cone, and the properties of the fluid in which the cone is rotating (density and viscosity). Ie

$$T = f(\omega, \rho, \mu, D) \qquad \text{[Eq 1]}$$

In order to fully investigate this system we would want to conduct a range of experiments with different shaft speed, density, viscosity and cone diameter. Suppose we take 10 values for each then we have to conduct 10^4 experiments. And even if we use some type of design of experiments approach to reduce the number of tests we need to do, we still end up doing a lot of testing at a huge cost both in terms of manufacture of the different models/rigs and testing time.

Using dimensional analysis it transpires that for this example we can represent the same data about how all these effects interact by forming two non-dimensional groups. These are a torque coefficient (also called moment coefficient), $Cm = \frac{T}{\frac{1}{2}\rho\omega^2 D^5}$ and a rotating Reynolds number $Re = \frac{\rho\omega D^2}{\mu}$ and thus reduce [Eq 1] to a *functional relationship*:

$$Cm = g(Re) \qquad \text{[Eq 2]}$$

Where the function *g* is not the same as the function *f*, but all the same information is contained.

In order to test this result I can plot the same data with Reynolds number on the x-axis and torque coefficient on the y-axis, as illustrated in Figure 3. Here the data now all collapses to a single line showing that the data is well represented by the dimensionless numbers. If the data had not collapsed to a single line then this would indicate that other factors were important but had not been considered.

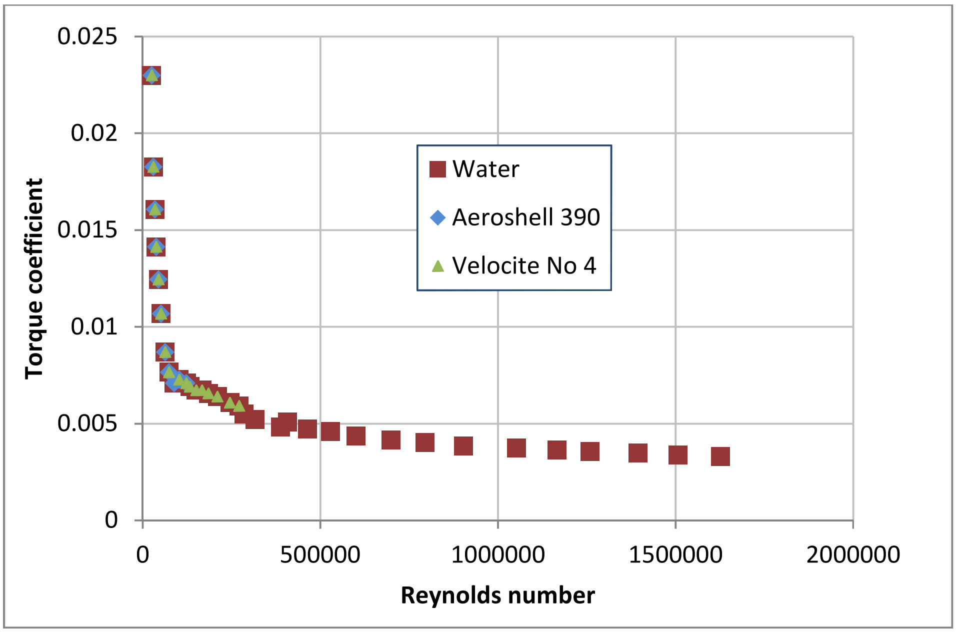

Figure 3: Non-dimensional form of the data from Figure 2

The keep points in this type of analysis lies in identifying which are they key variables in a system under investigation (for example we did not include cone angle, length of cone, cone surface finish, size of cone enclosure to name but a few) and deciding which non-dimensional groups to create. There is a method that supports this kind of analysis and this is called the Buckingham-Π theory (see later).

Another area where dimensional analysis is beneficial is when conducting tests with the actual fluids concerned may not be viable. For example it may be far safer and cheaper to run experiments at low speed using water than at high speed using air. And it is unlikely to be viable to run experiments with a strong acid or a highly volatile fuel. We can get the same Reynolds number at much lower rotation speeds if water is the working fluid compared to air. For example looking at the data on Figure 3, for the maximum plotted Re of around $1.6\text{x}10^6$, the shaft speed is 40.6 rad/s (388 rpm) for water and 1326 rad/s (12662 rpm) for air at 500K. If we know that we have only to match Reynolds number (and this is discussed later) then we might much prefer to do experiments with water than with hot, high speed air.

The main benefits of dimensional analysis can therefore be seen in:

- Geometrically scaling experiments
- Using different working fluids for experiments
- Reducing the number of experiments required
- Experimental/data insight

Worked Example 11

If the rotating cone of Figure 1 has a base diameter of 120mm and is to be rotated first in ambient air and subsequently in water. What rotational speed should be used in water to obtain the same Reynolds number as that obtained in air at 3000 rpm? Take the density and viscosity of air to be 1.2 kg/m^3 and $1.8\text{x}10^{-5}$ kg/ms respectively and the density and viscosity of water to be 1000 kg/m^3 and 0.001 kg/ms respectively.

Answer: 200 rpm

SOME "SELF EVIDENT TRUTHS"!

1. Dimensional homogeneity

"If an equation truly expresses a proper relationship between variables in a physical process it will be dimensionally homogeneous; that is each of its additive terms will have the same dimensions." [1]

So for example in Bernouill:

$$p + \frac{1}{2}\rho v^2 + \rho g z = constant$$

Each of the terms (including the constant) has the units of pressure and the dimensions of $[ML^{-1}T^{-2}]$. We make use of this principle in dimensional analysis.

2. Pure constants have no dimensions eg π, $\frac{1}{2}$, e
3. Angles are dimensionless.

THE BUCKINGHAM Π THEORY

There is a reasonably straightforward method for reducing a set of dimensional variables into a smaller set of non-dimensional groups. The method was first proposed in 1914 by a chap named Edgar Buckingham [**Error! Reference source not found.**] and π, in mathematical notation means a product of variables.

In its simplest form the Buckingham-Π theory says that if a process is fully described by *n* variables and *k* dimensions then we should be able to form m=*n-k* dimensionless groups (Π_1. Π_2 etc). For example suppose we have 5 variables that fully define our experiment or process such that:

$$v_1 = f(v_2, v_3, v_4, v_5)$$

and we further determine that the total number of dimensions is three, [M L T] say. Then n=5, k=3 and m=2 so we should be able to form two non-dimensional groups or Π's. This will give us a functional relationship:

$$\Pi_1 = g(\Pi_2)$$

Or in words, Π_1 is a function of Π_2.

We then pick out the two variables we are most interested in (the output variable v_1 and one of the other variables v_5) and use the other 3 to form the groups. It is important that these three do not themselves form a non-dimensional group. The two Πs are then formed by the variable we chose and power products of the three others. So:

$$\Pi_1 = \mathrm{v}_1(\mathrm{v}_2)^{\mathrm{a}}(\mathrm{v}_3)^{\mathrm{b}}(\mathrm{v}_4)^{\mathrm{c}}$$
$$\Pi_2 = \mathrm{v}_5(\mathrm{v}_2)^{\mathrm{d}}(\mathrm{v}_3)^{\mathrm{e}}(\mathrm{v}_4)^{\mathrm{f}}$$

But we also know that Π_1 and Π_2 are dimensionless so we know that

$$[\Pi_1] = [\mathrm{v}_1][\mathrm{v}_2]^{\mathrm{a}}[\mathrm{v}_3]^{\mathrm{b}}[\mathrm{v}_4]^{\mathrm{c}} = [\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0]$$
$$[\Pi_2] = [\mathrm{v}_5][\mathrm{v}_2]^{\mathrm{d}}[\mathrm{v}_3]^{\mathrm{e}}[\mathrm{v}_4]^{\mathrm{f}} = [\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0]$$

Although this might look complicated it really isn't as the following worked example illustrates.

Worked Example 12

The torque on the rotating cone of Figure 1 is related to the rotation speed, fluid properties and cone base diameter:

$$T = f(\omega, \rho, \mu, D)$$

Identify the appropriate non-dimensional groups.

Answer: Cm, Re

As Worked Example 12 shows, the method is straight forward but needs clear criterion for the selection of problem variables. We could have chosen different variables as the base variables and we would have obtained equally valid (but less recognizable) non-dimensional groups (see solutions to worked example 12). The main idea lies in building on existing fluid mechanics knowledge as well as just applying the method. Viscosity is often chosen as one of the variables of interest (ie a non-repeating variable) because the Reynolds number is one of the dominant dimensionless numbers in the governing equations of fluid mechanics.

Summarising the Buckingham-Π process [1]

1. List all the important variables associated with your process or experiment. If any important variables are missing the process will fail or at least be flawed. Count variables to establish *n*. Remember to count all the variables!
2. List the dimensions of each variable. Count the dimensions to establish k. A list of dimensions for common variables is given in**Error! Reference source not found.Error! Reference source not found.** (taken from [1]).
3. You should be able to form m=n-k Πs. Choose the variables you will use to form your Π-groups and ensure they are not themselves a non-dimensional group.
4. Set up your Π-groups with one of your independent variables and the repeating variables.
5. Balance exponents to find the form of each dimensionless group.

Worked Example 13

The power input to a centrifugal pump is a function of the volume flowrate, Q, impeller diameter D, rotation rate Ω and fluid properties μ and ρ.

$$P = f(Q, D, \Omega, \rho, \mu)$$

Re-write this as a dimensionless relationship using Ω, ρ and D as the repeating variables.

Answer: $\frac{P}{\rho\Omega^3 D^5} = f\left(\frac{\mu}{\rho\Omega D^2}, \frac{Q}{\Omega D^3}\right)$

What about dimensionless variables?

In some cases a variable that affects the outcome has no dimensions. Examples of this are: the number of blades on a wind turbine; or the angle of attack for an aerofoil. Suppose worked example 13 were modified so that the number of blades on the impeller was included as a separate variable (let's call it N). How would the resulting functional relationship be modified?

- The number of variables is increased from 6 to 7
- The number of dimensions remains the same at 3
- We therefore need an additional dimensionless group
- That "group" is N.
- The functional relationship becomes: $\frac{P}{\rho\Omega^3 D^5} = f\left(\frac{\mu}{\rho\Omega D^2}, \frac{\mathrm{Q}}{\Omega \mathrm{D}^3}, N\right)$

SIMILARITY AND MODEL TESTING

In the two worked examples above the list of relevant variables has been given. In an engineering experiment or test the list of important variables has to be identified and there is a lot of judgment involved usually. For example should surface tension be included? Is gravity important? Is wall roughness relevant? The selection of them requires clear understanding of fluid mechanics.

Once the relevant variables have been identified and manipulated into the relevant non-dimensional groups, the next phase is to achieve similarity between the model to be tested and the prototype or real world application. So if

$$\Pi_1 = \mathrm{f}(\Pi_2, \Pi_3 \ldots \Pi_\mathrm{m})$$

then all we need to do is to ensure all the non-dimensional groups have the same values in the model as the full-size application. Ie

> *"Flow conditions for a model test are completely similar if all relevant dimensionless groups have the same corresponding values for the model and the prototype."[1]*

Another way of looking at this is to think of geometric similarity, kinematic similarity, dynamic similarity and thermal similarity.

GEOMETRIC SIMILARITY

This is the most intuitively obvious kind of similarity and concerns length scales. All length scales must be the same. In geometric scaling, angles are preserved so 10° on the prototype is 10° on the model. Geometric scaling is illustrated in Figure 4 where full-size and model wing sections are compared.

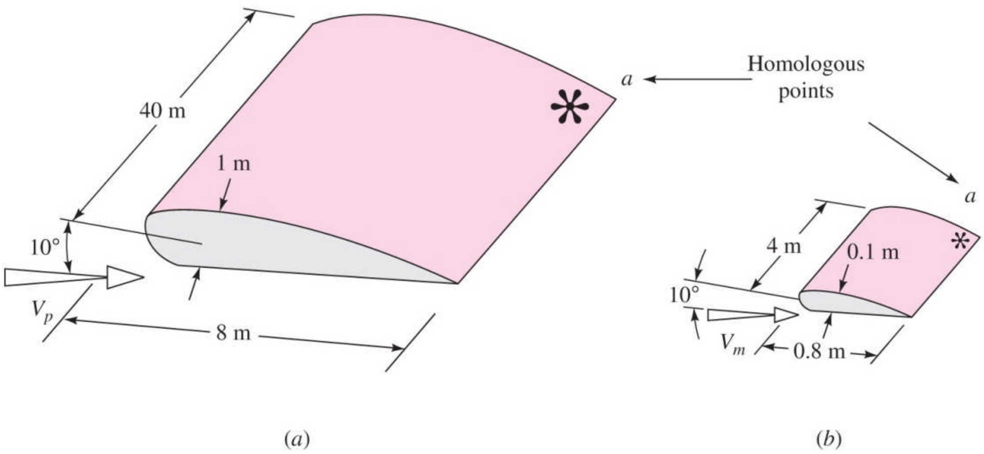

Figure 4: Illustrating geometric similarity between prototype and 1/10th scale model [1]

For geometric similarity the model must have:

a) Thickness, width and length 1/10th those of the prototype
b) Nose radius 1/10th that of prototype
c) Surface roughness 1/10th the size of that on the prototype
d) Any protruding fasteners (screws etc) should be of the same scale, in the same relative position (homologous) and protrude only 1/10th of full size.
e) Any coatings (eg paint) should be only 1/10th as thick.

In addition, prototype and model should have the same relative alignment to the incident flow (same angle of attack).
Some of these are not that easy to achieve in practice.

KINEMATIC SIMILARITY

In addition to geometric similarity, both model and prototype should have the same velocity scale ratio. Depending on what the particular case is, this is usually represented by Reynolds number, Mach number or Froude number. For example, frictionless free surface flows (potential flows) are kinematically similar if Froude numbers are equal.

$$Fr = \frac{U^2}{gL}$$

So we want

$$Fr_m = Fr_p$$

$$\frac{U_m^2}{gL_m} = \frac{U_p^2}{gL_p}$$

Geometric similarity gives us $\frac{L_p}{L_m} = R$ where R is the scale ratio.
Dynamic similarity in this case tells us that we must have

$$\frac{U_p^2}{U_m^2} = \frac{L_p}{L_m} = R$$

So

$$\frac{U_p}{U_m} = \sqrt{R}$$

In other words, for frictionless free surface flow, if the length scale ratio is 20:1 then the velocity scale ratio should be 4.47:1 so a length of 20m on the prototype is 1m on the model and a velocity of 20m/s on the prototype is 4.47m/s on the model.

DYNAMIC SIMILARITY

This is one step further, when model and prototype have the same length scale ratio, velocity scale ratio and force scale ratio. If geometric similarity exists, then kinematic and dynamic similarity are found if model and prototype force and pressure coefficients are identical. This occurs:

- For incompressible flow without free surface – when Reynolds numbers are equal
- For incompressible flow with free surface when Reynolds number and Froude number are equal. May also require Weber number and cavitation number to be equal depending on situation.
- For compressible flow Reynolds number, Mach number and specific heat ratio must be equal.

In some cases true dynamic similarity is almost impossible to achieve as Worked Example 14 illustrates.

Worked Example 14
A prototype boat is to be tested at a model scale of 50:1. This is an incompressible free surface flow and to ensure dynamic similarity both Reynolds number and Froude number must be equal. What kinematic viscosity must the model working fluid have if the prototype working fluid is water (ν=10^{-6} m^2/s) ?

Answer: $\nu_m = 2.83 \times 10^{-9}\ m^2/s$

A kinematic viscosity of 2.83×10^{-9} m^2/s is impossible to achieve, the closest liquid is mercury with $\nu = 1.16\times10^{-7}$ m^2/s. And mercury would not be a particularly good choice for lab testing!

The experimenter therefore has to make some compromises. Water is a good liquid for laboratory testing and in free surface flow, usually Froude number dominates. In this situation typically Froude number is held constant and model and prototype Reynolds numbers are allowed to differ. Using water instead of a higher density liquid gives lower Reynolds numbers for the model compared to the prototype and so data must be extrapolated, as illustrated in Figure 5. Extrapolation considerably increases the uncertainty in the data prediction.

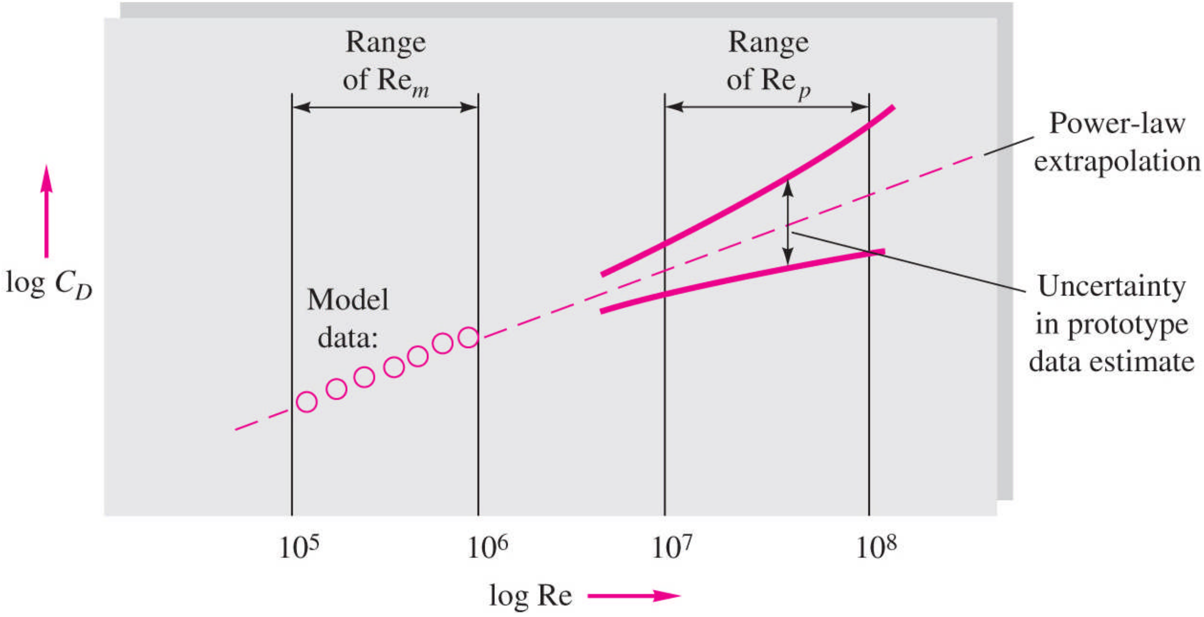

Figure 5: Reynolds number extrapolation

In real life, complete similarity is often not achieved and model scale experimental data must be interrogated carefully if spurious conclusions are not to be inferred. The main point is to identify the physical effect that want to be model and then select the corresponding similarity relation. A model does not represent the complete phenomenon but only the one that we are characterising on it.

REFERENCES

1. Frank White, Fluid Mechanics 5th Edition, 2003, ISBN: 0-07-119911-X, Chapter 5.
2. Yamada, Y., and Ito, M., 1979. \Frictional Resistance of Enclosed Rotating Cones With Superposed Through flow". *Journal of Fluids Engineering,* 101, June, pp. 259-264.

Note: there is plenty of references in the literature.

WORKED EXAMPLE SOLUTIONS

WORKED EXAMPLE 11

Rotating Reynolds number $Re = \frac{\rho\omega D^2}{\mu}$

Reynolds similarity: $Re_{air} = Re_{water}$

$$Re_{air} = \frac{1.2 \times \left(3000 \times \frac{2\pi}{60}\right) \times 0.12^2}{1.8 \times 10^{-5}} = 3.02 \times 10^5$$

$$Re_{water} = \frac{1000 \times \left(N \times \frac{2\pi}{60}\right) \times 0.12^2}{0.001} = 3.02 \times 10^5$$

N = 200 rpm

Relation between air and water rotational speed:

$$Re_{air} = \left(\frac{\rho\omega D^2}{\mu}\right)_{air} = Re_{water} = \left(\frac{\rho\omega D^2}{\mu}\right)_{water}$$

$$Re_{air} = \left(\frac{\rho N \left\{\frac{2\pi}{60}\right\} D^2}{\mu}\right)_{air} = Re_{water} = \left(\frac{\rho N \left\{\frac{2\pi}{60}\right\} D^2}{\mu}\right)_{water}$$

$$\left(\frac{\rho N}{\mu}\right)_{air} = \left(\frac{\rho N}{\mu}\right)_{water}$$

$$N_{water} = N_{air} \times \frac{\rho_{air}}{\rho_{water}} \frac{\mu_{water}}{\mu_{air}} = \frac{N_{air}}{15}$$

In the present case, with N_{air}= 3000 rpm then N_{water}= N = 200 rpm

WORKED EXAMPLE 12

$$T = f(\omega, \rho, \mu, D)$$

First find the dimensions of each variable:

Quantity	SI unit	Dimensions
ρ, density	kg/m^3	ML^{-3}
ω, angular velocity	rad/s	T^{-1}
D, length or diameter	m	L
μ, viscosity	kg/ms	$ML^{-1}T^{-1}$
T, torque	Nm	ML^2T^{-2}

There are 5 variables (n = 5) and 3 dimensions (k = 3) so we should be able to form 5-3=2 dimensionless groups. I choose my two variables as T and μ and this gives me:

$$[\Pi_1] = [T][\rho]^a[D]^b[\omega]^c = [M^0L^0T^0]$$
$$[\Pi_1] = [ML^2T^{-2}][ML^{-3}]^a[L]^b[T^{-1}]^c = [M^0L^0T^0]$$

The variables used to form the Π-groups are ρ, D and ω. These cannot be used to form a non-dimensional group so are suitable.

Remember when you multiply you add the exponents so that $x^a x^b = x^{a+b}$. We can now look at each of the dimensions in turn and balance the exponents.

So taking [M] for example:

$$M^1 M^a = M^0$$

M: $1 + a = 0$ so a = -1

L: $2 - 3a + b = 0$

T: $-2 - c = 0$ so c=-2

Thus b = -5 (from the L equation)

Therefore $\Pi_1 = T\rho^{-1}D^{-5}\omega^{-2} = \frac{T}{\rho\omega^2 D^5}$

Compare this to the torque coefficient mentioned above, $Cm = \frac{T}{\frac{1}{2}\rho\omega^2 D^5}$. The only difference is the half in the denominator. Dimensional analysis cannot identify constants.

We can follow the same process for the second non-dimensional group:

$$[\Pi_2] = [\mu][\rho]^d[D]^e[w]^f = [M^0L^0T^0]$$
$$[\Pi_2] = [ML^{-1}T^{-1}][ML^{-3}]^d[L]^e[T^{-1}]^f = [M^0L^0T^0]$$

Again balancing the exponents:

M: 1+d=0 so d=-1

L: -1-3d+e=0

T: -1-f=0 so f=-1

Thus e=-2

$$\Pi_2 = \mu\rho^{-1}D^{-2}\omega^{-1} = \frac{\mu}{\rho\omega D^2}$$

Compare this to the rotating Reynolds number $Re = \frac{\rho\omega D^2}{\mu}$ and we can see that Π_2 is 1/Re (which is also dimensionless of course). Recognising Reynolds number we would therefore take Re and Cm as our two non-dimensional groups.

The functional relationship is therefore *Cm=f(Re)*

Alternatively, we can select the following variables for the second non-dimensional group:

$$[\pi_2] = [\rho][\mu]^d[D]^e[W]^f = [M^0L^0T^0]$$

$$[\pi_2] = [ML^{-3}][ML^{-1}T^{-1}]^d[L]^e[T^{-1}]^f = [M^0L^0T^0]$$

$$M:\ 1 + d = 0;\quad d = -1$$

$$L:\ -3 - d + e = 0;\quad e = 2$$

$$T:\ -d - f = 0;\quad f = 1$$

$$[\pi_2] = [\rho][\mu]^{-1}[D]^2[W]$$

$$\pi_2 = \frac{\rho\varpi D^2}{\mu} = R_e$$

These two approaches are identically since both of then define the R_e.

WORKED EXAMPLE 12 WITH DIFFERENT VARIABLES CHOSEN

Suppose I choose my two non-repeating variables as ρ and ω - these are not good choices because neither is the output variable T, however following it through, this gives me repeating variables of T, D and μ

$$[\Pi_1] = [\rho][T]^a[D]^b[\mu]^c = [M^0L^0T^0]$$
$$[\Pi_1] = [ML^{-3}][ML^2T^{-2}]^a[L]^b[ML^{-1}T^{-1}]^c = [M^0L^0T^0]$$

① M: $1 + a + c = 0$
② L: $-3 + 2a + b - c = 0$
③ T: $-2a - c = 0$

Combining ① and ③ gives us $1 + a - 2a = 0 \quad \rightarrow \quad a = 1$

$$\therefore c = -2 \text{ and } b = -1$$

Thus $\Pi_1 = \frac{\rho T}{D\mu^2}$

Similarly, for ω:

$$[\Pi_2] = [\omega][T]^d[D]^e[\mu]^f = [M^0L^0T^0]$$
$$[\Pi_2] = [T^{-1}][ML^2T^{-2}]^d[L]^e[ML^{-1}T^{-1}]^f = [M^0L^0T^0]$$

① M: $d + f = 0$
② L: $2d + e - f = 0$
③ T: $-1 - 2d - f = 0$

Combining ① and ③ gives us: $-1 - d = 0 \quad \rightarrow \quad d = -1$

$$\therefore f = 1 \text{ and } e = 3$$

Thus $\Pi_2 = \frac{\omega D^3 \mu}{T}$

The functional relationship is: $\frac{\rho T}{D\mu^2} = f\left(\frac{\omega D^3\mu}{T}\right)$

Although you can see that this works dimensionally, neither of these is a recognised non-dimensional group and so this is not as satisfactory (or useful) a solution. **They do not correspond to the similarity groups of our governing equations.**

However, the dimensionless variables in the π theorem are linear independents. If in the last case $\Pi_1 = \frac{\rho T}{D\mu^2}$ and $\Pi_2 = \frac{\omega D^3\mu}{T}$, we let

$$\Pi_1 \text{ X } \Pi_2 = \frac{\rho\varpi D^2}{\mu} = R_e \text{ (Dynamic relevant)}$$

and

$$2\,\pi_1 x\,\pi_2^2 = \frac{2\rho\varpi D}{T} = \frac{1}{C_m} \quad \text{(As before)}$$

WORKED EXAMPLE 13

$$P = f(Q, D, \Omega, \rho, \mu)$$

First find the dimensions of each variable:

Quantity	SI unit	Dimensions
P, power	W	ML^2T^{-3}
Q, volume flowrate	m^3/s	L^3T^{-1}
ρ, density	kg/m^3	ML^{-3}
Ω, angular velocity	rad/s	T^{-1}
D, length or diameter	m	L
μ, viscosity	kg/ms	$ML^{-1}T^{-1}$

There are 6 variables (n = 5) and 3 dimensions (k = 3) so we should be able to form 6-3=3 dimensionless groups.

Objective of the experiment: to determine P as function of Q by a *Re* similarity.

I choose my three variables as P, Q and μ with Ω, ρ and D as the repeating variables. Check the repeating variables cannot be used to form a non-dimensional group.

$$\text{Can } [\rho]^a[D]^b[\Omega]^c = [M^0L^0T^0]?$$
$$\text{Can } [ML^{-3}]^a[L]^b[T^{-1}]^c = [M^0L^0T^0]?$$

Not unless c=0 so these variables are okay.

This gives me

$$[\Pi_1] = [P][\rho]^a[D]^b[\Omega]^c = [M^0L^0T^0]$$
$$[\Pi_1] = [ML^2T^{-3}][ML^{-3}]^a[L]^b[T^{-1}]^c = [M^0L^0T^0]$$

M: $1 + a = 0$ so a = -1

L: $2 - 3a + b = 0$

T: $-3 - c = 0$ so c=-3

Thus b = -5 (from the L equation)

Therefore $\Pi_1 = P\rho^{-1}D^{-5}\Omega^{-3} = \frac{P}{\rho\Omega^3D^5}$

We can follow the same process for the second non-dimensional group:

$$[\Pi_2] = [\mu][\rho]^d[D]^e[\Omega]^f = [M^0L^0T^0]$$
$$[\Pi_2] = [ML^{-1}T^{-1}][ML^{-3}]^d[L]^e[T^{-1}]^f = [M^0L^0T^0]$$

Again balancing the indices:

M: 1+d=0 so d=-1

L: -1-3d+e=0

T: -1-f=0 so f=-1

Thus e=-2

$$\Pi_2 = \mu\rho^{-1}D^{-2}\Omega^{-1} = \frac{\mu}{\rho\Omega D^2}$$

Compare this to the rotating Reynolds number $Re = \frac{\rho\Omega D^2}{\mu}$ and we can see that Π_2 is 1/Re (which is also dimensionless of course).

And finally for Π_3

$$[\Pi_3] = [Q][\rho]^g[D]^h[\Omega]^i = [M^0L^0T^0]$$
$$[\Pi_3] = [L^3T^{-1}][ML^{-3}]^g[L]^h[T^{-1}]^i = [M^0L^0T^0]$$

M: g=0

L: 3-3g+h=0 and as g=0 then h=-3

T: -1-i=0 so i=-1

$$\Pi_3 = Q\rho^0D^{-3}\Omega^{-1} = \frac{Q}{\Omega D^3}$$

So

$$\frac{P}{\rho\Omega^3D^5} = f\left(\frac{\mu}{\rho\Omega D^2}, \frac{Q}{\Omega D^3}\right)$$

As will be seen in the Turbomachinery topic, these are the standard non-dimensional groups used for correlating pump power.

WORKED EXAMPLE 14

$$Fr = \frac{U^2}{gL}$$

So we want

$$Fr_m = Fr_p$$
$$\frac{U_m^2}{gL_m} = \frac{U_p^2}{gL_p}$$

Geometric similarity gives us $\frac{L_p}{L_m} = 50$

Dynamic similarity in this case tells us that we must have

$$\frac{U_p^2}{U_m^2} = \frac{L_p}{L_m} = 50$$

So

$$\frac{U_p}{U_m} = \sqrt{50}$$

$$Re = \frac{\rho UL}{\mu} = \frac{UL}{\nu}$$
$$Re_p = Re_m$$

So

$$\frac{U_m L_m}{\nu_m} = \frac{U_p L_p}{\nu_p}$$
$$\frac{\nu_m}{\nu_p} = \frac{U_m}{U_p}\frac{L_m}{L_p} = \frac{1}{50\sqrt{50}} = 0.00283$$

So $\nu_m = 2.83 \times 10^{-9}\ m^2/s$

This is impossible, the closest liquid is mercury with $\nu = 1.16\times10^{-7}$ m^2/s. And mercury would not be a particularly good choice for lab testing.